

On Painlevé Related Functions Arising in Random Matrix Theory

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Abstract

In deriving large n probability distribution function of the rightmost eigenvalue from the classical Random Matrix Theory Ensembles, one is faced with the question of finding large n asymptotic of certain coupled set of functions. This paper presents some of these functions in a new light.

1 Introduction

In the study of Edgeworth type expansions for the limiting distribution of the rightmost eigenvalue from Gaussian Random Matrix Ensembles, we run into finding large n expansions of many key functions. For the Tracy-Widom distribution derivation, one needs the large n limits of these functions, and they can all be expressed in terms of the couple pair q and p , where q is the Hastings-McLeod solution to Painlevé II equation behaving at infinity as the Airy function. The frequency of these functions in the study of the largest eigenvalue of Gaussian and Laguerre Random Matrix Ensemble points to the necessity of a study of these functions in their own right. We hope this will shed a light into understanding some derivations related to this aspect of Random Matrix Theory and related field making use of such functions.

If one tries to read through a proof of an expansion relating various asymptotic functions, it's easy to get lost in translation. But if the related functions are well known, the reader will probably have a different experience and therefore a better understanding of the techniques and tools used for the derivation.

We present in this paper the derivations of those functions arising in the study of the largest eigenvalue for the Gaussian Ensemble of Random Matrix Theory in a hope of achieving our goal set above.

Before stating our results, there is a need to define our functions.

For a Gaussian ensemble of $n \times n$ matrices, the probability density that the eigenvalues lie in an infinitesimal intervals about the points $x_1 < \dots < x_n$ is given by

$$\mathbb{P}_{n,\beta}(x_1, \dots, x_n) = C_{n\beta} \exp \left(-\frac{\beta}{2} \sum_1^n x_j^2 \right) \prod_{j < k} |x_j - x_k|^\beta. \quad (1.1)$$

Where $\beta = 1$ corresponds to the Gaussian Orthogonal Ensemble (GOE_n), $\beta = 2$ corresponds to the Gaussian Unitary Ensemble (GUE_n), and $\beta = 4$ for the Gaussian Symplectic Ensemble (GSE_n).

(1.1) can also be represented as a determinant involving the variables x_i' s. For the simplest case $\beta = 2$, we have

$$\mathbb{P}_{n,2}(x_1, \dots, x_n) = \frac{1}{n!} (\det[\varphi_{j-1}(x_i)])^2 = \frac{1}{n!} \det[K_{n,2}(x_i, x_j)]_{i,j=1,\dots,n} \quad (1.2)$$

with

$$K_{n,2}(x, y) = \sum_{k=0}^{n-1} \varphi_k(x) \varphi_k(y) = \sqrt{\frac{n}{2}} \frac{\varphi_n(x) \varphi_{n-1}(y) - \varphi_{n-1}(x) \varphi_n(y)}{x - y} \quad (1.3)$$

and

$$\varphi_k(x) = \frac{1}{(2^k k! \sqrt{\pi})^{1/2}} H_k(x) e^{-x^2/2} \quad \text{with } H_k(x) \text{ the Hermite polynomials}$$

obtained by orthogonalizing the sequence $\{x^k, k = 0, \dots, n-1\}$ with respect to e^{-x^2} over \mathbb{R} . Using this representation, it can be shown that the probability distribution function of the largest eigenvalue λ_{max} is given by the Fredholm determinant of the operator with kernel $K_{n,2}$ acting on the set (t, ∞) ,

$$F_{n,2}(t) = \mathbb{P}(\lambda_{max} < t) = \det(I - K_{n,2}). \quad (1.4)$$

In finding the Edgeworth type expansion of $F_{n,2}$, one needs large n expansion of (1.3) or what it amount to, the large n expansion of φ_n . In [3], we derived the following expression.

Let the rescaling function τ be defined by,

$$\tau(x) = \sqrt{2(n+c)} + 2^{-\frac{1}{2}} n^{-\frac{1}{6}} x, \quad (1.5)$$

then

$$\begin{aligned} \varphi_n(\tau(x)) = n^{\frac{1}{6}} \left\{ \text{Ai}(X) + \frac{(2c-1)}{2} \text{Ai}'(X) n^{-\frac{1}{3}} + \left[(10c^2 - 10c + \frac{3}{2}) X \text{Ai}(X) \right. \right. \\ \left. \left. + X^2 \text{Ai}'(X) \right] \frac{n^{-\frac{2}{3}}}{20} + O(n^{-1}) \text{Ai}(X) \right\} \end{aligned} \quad (1.6)$$

and

$$\varphi_{n-1}(\tau(x)) = n^{\frac{1}{6}} \left\{ \text{Ai}(X) + \frac{(2c+1)}{2} \text{Ai}'(X) n^{-\frac{1}{3}} + \left[(10c^2 + 10c + \frac{3}{2}) X \text{Ai}(X) \right. \right.$$

$$+ X^2 \text{Ai}'(X) \left] \frac{n^{-\frac{2}{3}}}{20} + O(n^{-1}) \text{Ai}(X) \right\} \quad (1.7)$$

Ai being the Airy function. These two functions enable us to obtain the following expansion of the GUE kernel.

$$K_{n,2}(\tau(X), \tau(Y)) d\tau(X) = \tau' K_n(\tau(X), \tau(Y)) dX = \left\{ K_{\text{Ai}}(X, Y) - c \text{Ai}(X) \text{Ai}(Y) n^{-\frac{1}{3}} + \right. \\ \left. \frac{1}{20} \left[(X+Y) \text{Ai}'(X) \text{Ai}'(Y) - (X^2 + XY + Y^2) \text{Ai}(X) \text{Ai}(Y) + \right. \right. \\ \left. \left. \frac{-20c^2 + 3}{2} (\text{Ai}'(X) \text{Ai}(Y) + \text{Ai}(X) \text{Ai}'(Y)) \right] n^{-\frac{2}{3}} + O(n^{-1}) E(X, Y) \right\} dX. \quad (1.8)$$

In deriving the finite but large n probability distribution function of the largest eigenvalue using (1.8), and representation (1.4) we have to factor out of (1.8) the constant term (with respect to n) to obtain the representation

$$F_{n,2}(\tau(t)) = \det \left((I - K_{\text{Ai}}(X, Y)) \cdot \left\{ I + (I - K_{\text{Ai}}(X, Y))^{-1} \left[c \text{Ai}(X) \text{Ai}(Y) n^{-\frac{1}{3}} - \right. \right. \right. \\ \left. \left. \frac{1}{20} \left[(X+Y) \text{Ai}'(X) \text{Ai}'(Y) - (X^2 + XY + Y^2) \text{Ai}(X) \text{Ai}(Y) + \right. \right. \right. \\ \left. \left. \left. \frac{-20c^2 + 3}{2} (\text{Ai}'(X) \text{Ai}(Y) + \text{Ai}(X) \text{Ai}'(Y)) \right] n^{-\frac{2}{3}} + O(n^{-1}) E(X, Y) \right] \right\} \right). \quad (1.9)$$

This Fredholm determinant is computed over the set (t, ∞) . Thus to complete the determination of $F_{n,2}(\tau(t))$ we need to determine the action of the integral operator $(I - K_{\text{Ai}})$ on $x^i \text{Ai}(x)$ and $x^i \text{Ai}'(x)$ where $i = 0, 1, \dots$. These are the special functions in the GUE case, and they are independent of n , they are well known in the literature (see for example [2, 3, 4, 20, 19, 17, 21, 23, 18, 22]). For these n independent functions, we just redefine them here and then introduce their n dependent counterparts.

$$K_{\text{Ai}}(X, Y) = \frac{\text{Ai}(X) \text{Ai}'(Y) - \text{Ai}(Y) \text{Ai}'(X)}{X - Y} = \int_0^\infty \text{Ai}(X+Z) \text{Ai}(Y+Z) dZ. \quad (1.10)$$

$$\rho(X, Y; s) = (I - K_{\text{Ai}})^{-1}(X, Y; s), \quad R(X, Y; s) = \rho(X, Y; s) \cdot K_{\text{Ai}}(X, Y) \quad (1.11)$$

this last product is operator multiplication.

$$Q_i(x; s) = ((I - K_{\text{Ai}})^{-1}, x^i \text{Ai}), \quad (1.12)$$

$$P_i(x; s) = ((I - K_{\text{Ai}})^{-1}, x^i \text{Ai}'), \quad (1.13)$$

$$q_i(s) = Q_i(s; s), \quad p_i(s) = P_i(s; s), \quad q_0(s) := q(s) \quad p_0(s) := p(s) \quad (1.14)$$

$$u_i(s) = (Q_i, \text{Ai}), \quad v_i(s) = (P_i, \text{Ai}), \quad u_0(s) := u(s), \quad v_0(s) := v(s) \quad (1.15)$$

$$\tilde{v}_i(s) = (Q_i, \text{Ai}'), \quad w_i(t) = (P_i, \text{Ai}'), \quad w_0(s) := w(s), \quad \text{and} \quad \tilde{v}_0(s) := \tilde{v}(s). \quad (1.16)$$

Here (\cdot, \cdot) denotes the inner product on $L^2(s, \infty)$ and $i = 0, 1, 2, \dots$. These are all well known functions, this paper is concerned with the n dependent counterparts whose definitions are similar in nature. the changes needed here are on the kernel definition. The operator kernel is of the same form as (1.8)

$$K_n(x, y) = \frac{\varphi(x)\psi(y) - \psi(x)\varphi(y)}{x - y} \quad (1.17)$$

with

$$\varphi(x) = \sqrt[4]{\frac{n}{2}} \varphi_n(x) \quad \text{and} \quad \psi(x) = \sqrt[4]{\frac{n}{2}} \varphi_{n-1}(x)$$

Relating this to the previous set of function are the functions φ and ψ , they are Ai and Ai'. We have the following functions,

$$\rho_n(x, y; t) := (I - K_n)^{-1}(x, y; t), \quad R_n(x, y; t) := \int_t^\infty \rho_n(x, z; t) K_n(z, y; t) dz \quad (1.18)$$

these are kernels of integral operators on (t, ∞)

$$Q_{n,i}(x; t) := \int_t^\infty \rho_n(x, y; t) y^i \varphi(y) dy, \quad P_{n,i}(x; t) := \int_t^\infty \rho_n(x, y; t) y^i \psi(y) dy \quad (1.19)$$

or

$$Q_n(x; t) := (\rho_n, \varphi)_{(t, \infty)} \quad P_n(x; t) := (\rho_n, \psi)_{(t, \infty)}.$$

And the other functions are

$$q_{n,i}(t) = Q_{n,i}(t; t), \quad p_{n,i}(t) = P_{n,i}(t; t), \quad q_{n,0}(t) := q_n(t) \quad p_{n,0}(s) := p_n(t) \quad (1.20)$$

$$u_{n,i}(t) = (Q_{n,i}, \varphi), \quad v_{n,i}(t) = (P_{n,i}, \varphi), \quad u_{n,0}(t) := u_n(t), \quad v_{n,0}(t) := v_n(t) \quad (1.21)$$

$$\tilde{v}_{n,i}(t) = (Q_{n,i}, \psi), \quad w_{n,i}(t) = (P_{n,i}, \psi), \quad w_{n,0}(t) := w_n(t), \quad \text{and} \quad \tilde{v}_{n,0}(t) := \tilde{v}_n(t). \quad (1.22)$$

Here (\cdot, \cdot) denotes the inner product on $L^2(t, \infty)$ and $i = 0, 1, 2, \dots$.

We will like to point out the following ambiguity in these definitions, the n -independent functions have a subscript i whereas the n dependent ones have the subscript n . We were not able to find a suitable representations of the set of functions depending on the matrix ensemble of $n \times n$ matrices, but the choice of keeping with the original Tracy and widom notation was made in part to help the reader go through the topic without too much confusion. Thus whenever we use the subscript n we will refer to the large size on the underlying matrix ensemble and when i is used it refers to the exponent of the variable x appearing in the definition of that specific function and i takes values from $0, 1, 2, \dots$. One exception is when we will use a second subscript to distinguish between the 3 beta ensembles $\beta = 1, 2, 4$, in this case we will remind the reader of the significance of those values.

In deriving the probability distribution function of the largest eigenvalue $F_{n,1}(t)$ for the orthogonal ensemble, and $F_{n,4}(t)$ for the symplectic ensemble, we encounter new sets on functions obeying the same set of relations.

If we define ε to be the integral operator with kernel $\varepsilon(x, y) = \frac{1}{2}\text{sign}(x - y)$ then

$$Q_{n,\varepsilon}(x; t) := \int_t^\infty \rho_n(x, y; t) \varepsilon(\varphi)(y) dy, \quad q_{n,\varepsilon}(t) := Q_{n,\varepsilon}(t; t) \quad (1.23)$$

$$P_{n,\varepsilon}(x; t) := \int_t^\infty \rho_n(x, y; t) \varepsilon(\psi)(y) dy, \quad p_{n,\varepsilon}(t) := P_{n,\varepsilon}(t; t). \quad (1.24)$$

In a similar way we define

$$u_{n,\varepsilon}(t) := \int_t^\infty Q_{n,\varepsilon}(x; t) \varphi(x) dx, \quad v_{n,\varepsilon}(t) := \int_t^\infty P_{n,\varepsilon}(x; t) \varphi(x) dx. \quad (1.25)$$

$$\tilde{v}_{n,\varepsilon}(t) := \int_t^\infty Q_{n,\varepsilon}(x; t) \psi(x) dx, \quad \text{and} \quad w_{n,\varepsilon}(t) := \int_t^\infty P_{n,\varepsilon}(x; t) \psi(x) dx. \quad (1.26)$$

And finally we also have for the Gaussian Orthogonal Ensemble

$$\mathcal{R}_{n,1}(t) := \int_{-\infty}^t R_n(x, t; t) dx, \quad \mathcal{P}_{n,1}(t) := \int_{-\infty}^t P_n(x; t) dx, \quad \mathcal{Q}_{n,1}(t) := \int_{-\infty}^t Q_n(x; t) dx \quad (1.27)$$

(Note here that the second subscript here refers to the beta being 1 for the orthogonal ensemble and has nothing to do with the previous discussion on i and n .) For the Gaussian Symplectic Ensemble we have,

$$\mathcal{R}_{n,4}(t) := \int_{-\infty}^\infty \varepsilon(x, t) R_n(x, t; t) dx, \quad \mathcal{P}_{n,4}(t) = \int_{-\infty}^\infty \varepsilon(x - t) P_n(x; t) dx, \quad \text{and} \quad (1.28)$$

$$\mathcal{Q}_{n,4}(t) := \int_{-\infty}^\infty \varepsilon(x - t) Q_n(x; t) dx,$$

and the 4 refers to beta being 4 for the Gaussian Symplectic Ensemble.

We have the large n expansion of most of these functions from previous work. What is new in this paper are the large n expansion of $Q_{n,i}$, $P_{n,i}$ this can be used to derive an expansion for $u_{n,i}$, $v_{n,i}$, $\tilde{v}_{n,i}$, $w_{n,i}$. We also have closed formula for $u_{n,\varepsilon}$, $\tilde{v}_{n,\varepsilon}$, $q_{n,\varepsilon}$, $\mathcal{Q}_{n,1}$, $\mathcal{P}_{n,1}$, $\mathcal{R}_{n,1}$, $\mathcal{Q}_{n,4}$, $\mathcal{P}_{n,4}$, and $\mathcal{R}_{n,4}$.

In the second section we will give a brief justification of $Q_{n,i}$ and $P_{n,i}$ follow in the third section with the justification of these last 9 functions. Again the motivation for the derivation of these functions is due to their appearance in the Edgeworth type expansion of the largest eigenvalue probability distribution function for the Gaussian Orthogonal and Symplectic Ensembles.

2 Epsilon independent functions

Building on (2.5), (1.7) and (1.8) we find that

$$Q_{n,i}(x) := ((I - K_{n,2})^{-1}(x, y; t), y^i \varphi(y)) = \int_t^\infty (I - K_{n,2})^{-1}(x, y; t) y^i \varphi(y) dy$$

$$P_{n,i}(x) := ((I - K_{n,2})^{-1}(x, y; t), y^i \psi(y)) = \int_t^\infty (I - K_{n,2})^{-1}(x, y; t) y^i \psi(y) dy$$

therefore we need to find $\rho_n(x, y; t) = (I - K_{n,2})^{-1}(x, y; t)$ in order to find an expressions for these two functions. But

$$\begin{aligned} (I - K_{n,2})^{-1}(\tau(X), \tau(Y); \tau(t)) &= \left\{ I + (I - K_{\text{Ai}})^{-1}(X, Y; t) \left[c \text{Ai}(X) \text{Ai}(Y) n^{-\frac{1}{3}} - \right. \right. \\ &\quad \left. \frac{1}{20} \left[(X + Y) \text{Ai}'(X) \text{Ai}'(Y) - (X^2 + XY + Y^2) \text{Ai}(X) \text{Ai}(Y) + \right. \right. \\ &\quad \left. \left. \frac{-20c^2 + 3}{2} (\text{Ai}'(X) \text{Ai}(Y) + \text{Ai}(X) \text{Ai}'(Y)) \right] n^{-\frac{2}{3}} + O(n^{-1}) E(X, Y) \right] \Big\}^{-1} \cdot (I - K_{\text{Ai}}(X, Y))^{-1} \\ &= \left\{ I + cQ(X) \text{Ai}(Y) n^{-\frac{1}{3}} - \frac{1}{20} \left[(P_1(X) + YP(X)) \text{Ai}'(Y) - \right. \right. \\ &\quad \left. \left. (Q_2(X) + YQ_1(X) + Y^2Q(X)) \text{Ai}(Y) + \frac{-20c^2 + 3}{2} (P(X) \text{Ai}(Y) + Q(X) \text{Ai}'(Y)) \right] n^{-\frac{2}{3}} \right. \\ &\quad \left. + O\left(\frac{1}{n}\right) E(X, Y) \right] \Big\}^{-1} \cdot (I - K_{\text{Ai}}(X, Y))^{-1} \\ &= \left\{ I - cQ(X) \text{Ai}(Y) n^{-\frac{1}{3}} + \frac{1}{20} \left[(P_1(X) + YP(X)) \text{Ai}'(Y) - \right. \right. \\ &\quad \left. \left. (Q_2(X) + YQ_1(X) + Y^2Q(X)) \text{Ai}(Y) + \frac{-20c^2 + 3}{2} (P(X) \text{Ai}(Y) + Q(X) \text{Ai}'(Y)) \right. \right. \\ &\quad \left. \left. + 20c^2 Q(X; s) u(s) \text{Ai}(Y) \right] n^{-\frac{2}{3}} + O\left(\frac{1}{n}\right) E(X, Y) \right] \Big\} \cdot (I - K_{\text{Ai}}(X, Y))^{-1} = \rho_n(X, Y; t) \end{aligned} \tag{2.1}$$

Note that with this representation of ρ all the $Q_{n,i}$ and $P_{n,i}$ will have no term independent of n , but only $Q_{n,0} := Q_n$ and $P_{n,0} := P_n$, in [3] we find that

$$\begin{aligned} Q_n(\tau(X); \tau(s)) &= n^{\frac{1}{6}} \left[Q(X; s) + \left[\frac{2c - 1}{2} P(X; s) - cQ(X; s) u(s) \right] n^{-\frac{1}{3}} \right. \\ &\quad + \left[(10c^2 - 10c + \frac{3}{2}) Q_1(X; s) + P_2(X; s) + (-30c^2 + 10c + \frac{3}{2}) Q(X; s) v(s) \right. \\ &\quad \left. + P_1(X; s) v(s) + P(X; s) v_1(s) - Q_2(X; s) u(s) - Q_1(X; s) u_1(s) - Q(X; s) u_2(s) \right] \end{aligned}$$

$$+ (-10c^2 + \frac{3}{2})P(X; s)u(s) + 20c^2Q(X; s)u^2(s) \Big] \frac{n^{-\frac{2}{3}}}{20} + O(n^{-1})E_q(X; s) \Big], \quad (2.2)$$

and

$$\begin{aligned} P_n(\tau(X); \tau(s)) &= n^{\frac{1}{6}} \left[Q(X; s) + \left[\frac{2c+1}{2}P(X; s) - cQ(X; s)u(s) \right] n^{-\frac{1}{3}} \right. \\ &+ \left[(10c^2 + 10c + \frac{3}{2})Q_1(X; s) + P_2(X; s) + (-30c^2 - 10c + \frac{3}{2})Q(X; s)v(s) \right. \\ &+ P_1(X; s)v(s) + P(X; s)v_1(s) - Q_2(X; s)u(s) - Q_1(X; s)u_1(s) - Q(X; s)u_2(s) \\ &\left. + (-10c^2 + \frac{3}{2})P(X; s)u(s) + 20c^2Q(X; s)u^2(s) \right] \frac{n^{-\frac{2}{3}}}{20} + O(n^{-1})E_p(X; s) \Big]. \quad (2.3) \end{aligned}$$

Using

$$\begin{aligned} Q_{n,i}(\tau(X), \tau(s)) &= (\rho_n(\tau(X), \tau(Y), \tau(s)), (\tau(Y))^i \varphi(\tau(Y)))_{(\tau(s) \rightarrow \infty)} = \\ &\sum_{k=0}^i \frac{i!}{k! (i-k)!} \frac{2^{\frac{i}{2}}(n+c)^{\frac{i-k}{2}}}{2^k n^{\frac{k}{2}}} (\rho_n(\tau(X), \tau(Y); \tau(s)), Y^k \varphi(\tau(Y)))_{(\tau(s) \rightarrow \infty)}. \end{aligned}$$

and

$$\begin{aligned} X^k \varphi(\tau(X)) &= n^{\frac{1}{6}} \left\{ X^k \text{Ai}(X) + \frac{(2c-1)}{2} X^k \text{Ai}'(X) n^{-\frac{1}{3}} + \left[(10c^2 - 10c + \frac{3}{2}) X^{k+1} \text{Ai}(X) \right. \right. \\ &\left. \left. + X^{k+2} \text{Ai}'(X) \right] \frac{n^{-\frac{2}{3}}}{20} + O(n^{-1}) \text{Ai}(X) \right\} \quad (2.4) \end{aligned}$$

we find that

$$\begin{aligned} \rho(X, Y; s) \cdot \varphi(\tau(X)) &= n^{\frac{1}{6}} \left\{ Q_k(X; s) + \frac{(2c-1)}{2} P_k(X) n^{-\frac{1}{3}} + \left[(10c^2 - 10c + \frac{3}{2}) Q_{k+1}(X) \right. \right. \\ &\left. \left. + P_{k+2}(X) \right] \frac{n^{-\frac{2}{3}}}{20} + O(n^{-1}) Q_k(X) \right\}. \quad (2.5) \end{aligned}$$

Combining this with the action of the first factor on the right of (2.1) gives the following expression for $(\rho_n(\tau(X), \tau(Y); \tau(s)), X^k \varphi(\tau(X)))$

$$\begin{aligned} n^{\frac{1}{6}} \left\{ Q_k(X; s) + \left[\frac{2c-1}{2} P_k(X; s) - cu_k(s)Q(X; s) \right] n^{-\frac{1}{3}} + [(10c - 20c^2)\tilde{v}_k(s)Q(X; s) + \right. \\ (10c^2 - 10c + \frac{3}{2})Q_{k+1}(X; s) + P_{k+2}(X; s) + P_1(X; s)v_k(s) + P(X)(Y \text{Ai}'(Y), Q_k(Y))_{(s \rightarrow \infty)} \\ - u_k(s)Q_2(X; s) - Q_1(X; s)(Y \text{Ai}(Y), Q_k(Y; s))_{(s \rightarrow \infty)} - Q(X; s)(Y^2 \text{Ai}(Y), Q_k(Y; s))_{(s \rightarrow \infty)} \\ \left. + \frac{-20c^2 + 3}{2} P(X; s)u_k(s) + \frac{-20c^2 + 3}{2} Q(X; s)v_k(s) + 20c^2 Q(X; s)u(s)u_k(s) \right] \frac{n^{-\frac{2}{3}}}{20} \end{aligned}$$

$$+O\left(\frac{1}{n}\right)E(X; s)\Big\}.$$

To simplify the inner product in this last expression, we use the following recurrence relation derived in [21]

$Q_k(X; s) = X^k Q(X; s) - \sum_{i+j=k-1; i, j > 0} (v_j Q_i - u_j P_i)$ to have

$$\begin{aligned} (Q_k(X; s), X \text{Ai}(X))_{(s \rightarrow \infty)} &= \int_s^\infty \int_s^\infty X \text{Ai}(X) \rho(X, Y; s) Y^k \text{Ai}(Y) dY dX \\ &= (Q_1(X; s), X^k \text{Ai}(X)) = (XQ(X; s) + u(s)P(X; s) - v(s)Q(X; s), X^k \text{Ai}(X)) = \\ &\quad u_{k+1}(s) + u(s)\tilde{v}_k(s) - v(s)u_k(s) \end{aligned}$$

and

$$\begin{aligned} (Q_k(X; s), X^2 \text{Ai}(X))_{(s \rightarrow \infty)} &= (Q_2(X; s), X^k \text{Ai}(X))_{(s \rightarrow \infty)} = \\ &= (X^2 Q(X; s) - v(s)(XQ(X; s) + u(s)P(X; s) - v(s)Q(X; s)) - u(s)(XP(X; s) \\ &\quad - w(s)Q(X; s) + v(s)P(X; s)) - v_1(s)Q(X; s) + u_1(s)P(X; s), X^k \text{Ai}(X)) \\ &= u_{k+2}(s) - v(s)u_{k+1} - v(s)u(s)\tilde{v}_k(s) + v(s)^2 u_k(s) - u(s)\tilde{v}_{k+1}(s) \\ &\quad + u(s)w(s)u_k(s) - u(s)v(s)\tilde{v}_k(s) - v_1(s)u_k(s) + u_1(s)\tilde{v}_k(s). \end{aligned}$$

we also have

$$\begin{aligned} (Q_k(X; s), X \text{Ai}'(X))_{(s \rightarrow \infty)} &= (P_1(X; s), X^k \text{Ai}(X)) = \\ &\quad \tilde{v}_{k+1}(s) + v(s)\tilde{v}_k(s) - w(s)u_k(s). \end{aligned}$$

We therefore have

$$\begin{aligned} Q_{n,i}(\tau(X), \tau(s)) &= \sum_{k=0}^i \frac{i!}{k! (i-k)!} \frac{2^{\frac{i}{2}-k} (n+c)^{\frac{i-k}{2}}}{n^{\frac{k}{2}-\frac{1}{6}}} \cdot \left\{ Q_k(X; s) + \right. \\ &\quad \left[\frac{2c-1}{2} P_k(X; s) - cu_k(s)Q(X; s) \right] n^{-\frac{1}{3}} + \left[(10c-20c^2)\tilde{v}_k(s)Q(X; s) + \right. \\ &\quad \left. (10c^2-10c+\frac{3}{2})Q_{k+1}(X; s) + P_{k+2}(X; s) + P_1(X; s)v_k(s) + P(X) \left(\tilde{v}_{k+1}(s) + v(s)\tilde{v}_k(s) - w(s)u_k(s) \right) \right. \\ &\quad \left. - u_k(s)Q_2(X; s) - Q_1(X; s) \left(u_{k+1}(s) + u(s)\tilde{v}_k(s) - v(s)u_k(s) \right) \right. \\ &\quad \left. - Q(X; s) \left(u_{k+2}(s) - v(s)u_{k+1} - v(s)u(s)\tilde{v}_k(s) + v(s)^2 u_k(s) - u(s)\tilde{v}_{k+1}(s) \right. \right. \\ &\quad \left. \left. + u(s)w(s)u_k(s) - u(s)v(s)\tilde{v}_k(s) - v_1(s)u_k(s) + u_1(s)\tilde{v}_k(s) \right) \right. \\ &\quad \left. + \frac{-20c^2+3}{2} P(X; s)u_k(s) + \frac{-20c^2+3}{2} Q(X; s)v_k(s) + 20c^2 Q(X; s)u(s)u_k(s) \right] \frac{n^{-\frac{2}{3}}}{20} \end{aligned}$$

$$+O\left(\frac{1}{n}\right)E(X; s) \Big\}. \quad (2.6)$$

In a similar way we have

$$P_{n,i}(\tau(X), \tau(s)) = (\rho_n(\tau(X), \tau(Y), \tau(s)), (\tau(Y))^i \psi(\tau(Y)))_{(\tau(s) \rightarrow \infty)} = \\ \sum_{k=0}^i \frac{i!}{k! (i-k)!} \frac{2^{\frac{i}{2}} (n+c)^{\frac{i-k}{2}}}{2^k n^{\frac{k}{2}}} (\rho_n(\tau(X), \tau(Y); \tau(s)), Y^k \psi(\tau(Y)))_{(\tau(s) \rightarrow \infty)}.$$

And $(\rho_n(\tau(X), \tau(Y); \tau(s)), X^k \psi(\tau(X)))$ is equal to

$$n^{\frac{1}{6}} \left\{ Q_k(X; s) + \left[\frac{2c+1}{2} P_k(X; s) - cu_k(s) Q(X; s) \right] n^{-\frac{1}{3}} + [-(10c+20c^2) \tilde{v}_k(s) Q(X; s) + \right. \\ (10c^2+10c+\frac{3}{2}) Q_{k+1}(X; s) + P_{k+2}(X; s) + P_1(X; s) v_k(s) + P(X)(Y \text{Ai}'(Y), Q_k(Y))_{(s \rightarrow \infty)} \\ \left. - u_k(s) Q_2(X; s) - Q_1(X; s)(Y \text{Ai}(Y), Q_k(Y; s))_{(s \rightarrow \infty)} - Q(X; s)(Y^2 \text{Ai}(Y), Q_k(Y; s))_{(s \rightarrow \infty)} \right. \\ \left. + \frac{-20c^2+3}{2} P(X; s) u_k(s) + \frac{-20c^2+3}{2} Q(X; s) v_k(s) + 20c^2 Q(X; s) u(s) u_k(s) \right] \frac{n^{-\frac{2}{3}}}{20} \\ \left. + O\left(\frac{1}{n}\right) E(X; s) \right\}.$$

this therefore gives

$$P_{n,i}(\tau(X), \tau(s)) = \sum_{k=0}^i \frac{i!}{k! (i-k)!} \frac{2^{\frac{i}{2}-k} (n+c)^{\frac{i-k}{2}}}{n^{\frac{k}{2}-\frac{1}{6}}} \times \\ \left\{ Q_k(X; s) + \left[\frac{2c+1}{2} P_k(X; s) - cu_k(s) Q(X; s) \right] n^{-\frac{1}{3}} + [-(10c+20c^2) \tilde{v}_k(s) Q(X; s) + \right. \\ (10c^2+10c+\frac{3}{2}) Q_{k+1}(X; s) + P_{k+2}(X; s) + P_1(X; s) v_k(s) + \\ P(X) \left(\tilde{v}_{k+1}(s) + v(s) \tilde{v}_k(s) - w(s) u_k(s) \right) \\ - u_k(s) Q_2(X; s) - Q_1(X; s) \left(u_{k+1}(s) + u(s) \tilde{v}_k(s) - v(s) u_k(s) \right) \\ - Q(X; s) \left(u_{k+2}(s) - v(s) u_{k+1} - v(s) u(s) \tilde{v}_k(s) + v(s)^2 u_k(s) - u(s) \tilde{v}_{k+1}(s) \right. \\ \left. + u(s) w(s) u_k(s) - u(s) v(s) \tilde{v}_k(s) - v_1(s) u_k(s) + u_1(s) \tilde{v}_k(s) \right) \\ \left. + \frac{-20c^2+3}{2} P(X; s) u_k(s) + \frac{-20c^2+3}{2} Q(X; s) v_k(s) + 20c^2 Q(X; s) u(s) u_k(s) \right] \frac{n^{-\frac{2}{3}}}{20} \\ \left. + O\left(\frac{1}{n}\right) E(X; s) \right\}.$$

$$+O\left(\frac{1}{n}\right)E(X; s)\Big\}. \quad (2.7)$$

When we set i to zero we recover $Q_n(X; s)$ and $P_n(X; s)$.

We see immediately that these two series representations of $Q_{n,i}$ and $P_{n,i}$ are not in terms of $n^{-\frac{1}{3}}$ when i is not zero. We can use (2.5), (1.7), (2.6) and (2.9), to derive an expansion for $u_{n,i}$, $v_{n,i}$, $\tilde{v}_{n,i}$ and $w_{n,i}$ from their representations

$$u_{n,i}(t) = (Q_{n,i}(x; t), \varphi(x))_{(t \rightarrow \infty)} \quad v_{n,i}(t) = (P_{n,i}(x; t), \varphi(x))_{(t \rightarrow \infty)}$$

$$\tilde{v}_{n,i}(t) = (Q_{n,i}(x; t), \psi(x))_{(t \rightarrow \infty)} \quad \text{and} \quad w_{n,i}(t) = (P_{n,i}(x; t), \psi(x))_{(t \rightarrow \infty)}.$$

We would like to note that (2.6) and (2.9) are the new quantities in this section, as additional corollary the derivation of $q_{n,i}(t)$ and $p_{n,i}(t)$.

$$\begin{aligned} q_{n,i}(\tau(s)) = & \sum_{k=0}^i \frac{i!}{k! (i-k)!} \frac{2^{\frac{i}{2}-k} (n+c)^{\frac{i-k}{2}}}{n^{\frac{k}{2}-\frac{1}{6}}} \cdot \left\{ q_k(s) + \right. \\ & \left[\frac{2c-1}{2} p_k(s) - cu_k(s)q(s) \right] n^{-\frac{1}{3}} + \left[(10c-20c^2)\tilde{v}_k(s)q(s) + \right. \\ & (10c^2-10c+\frac{3}{2})q_{k+1}(s) + p_{k+2}(s) + p_1(s)v_k(s) + p(s) \left(\tilde{v}_{k+1}(s) + v(s)\tilde{v}_k(s) - w(s)u_k(s) \right) \\ & - u_k(s)q_2(s) - q_1(s) \left(u_{k+1}(s) + u(s)\tilde{v}_k(s) - v(s)u_k(s) \right) \\ & - q(s) \left(u_{k+2}(s) - v(s)u_{k+1} - v(s)u(s)\tilde{v}_k(s) + v(s)^2u_k(s) - u(s)\tilde{v}_{k+1}(s) \right) \\ & \left. + u(s)w(s)u_k(s) - u(s)v(s)\tilde{v}_k(s) - v_1(s)u_k(s) + u_1(s)\tilde{v}_k(s) \right) \\ & \left. + \frac{-20c^2+3}{2} p(s)u_k(s) + \frac{-20c^2+3}{2} Q(X; s)v_k(s) + 20c^2q(s)u(s)u_k(s) \right] \frac{n^{-\frac{2}{3}}}{20} \\ & \left. + O\left(\frac{1}{n}\right)e_q(s) \right\}, \quad (2.8) \end{aligned}$$

$$\begin{aligned} p_{n,i}(\tau(s)) = & \sum_{k=0}^i \frac{i!}{k! (i-k)!} \frac{2^{\frac{i}{2}-k} (n+c)^{\frac{i-k}{2}}}{n^{\frac{k}{2}-\frac{1}{6}}} \times \\ & \left\{ q_k(s) + \left[\frac{2c+1}{2} p_k(s) - cu_k(s)q(s) \right] n^{-\frac{1}{3}} + \left[-(10c+20c^2)\tilde{v}_k(s)q(s) + \right. \right. \\ & (10c^2+10c+\frac{3}{2})q_{k+1}(s) + p_{k+2}(s) + p_1(s)v_k(s) + \\ & \left. \left. p(s) \left(\tilde{v}_{k+1}(s) + v(s)\tilde{v}_k(s) - w(s)u_k(s) \right) \right] \right\} \end{aligned}$$

$$\begin{aligned}
& -u_k(s)q_2(s) - q_1(s)\left(u_{k+1}(s) + u(s)\tilde{v}_k(s) - v(s)u_k(s)\right) \\
& -q(s)\left(u_{k+2}(s) - v(s)u_{k+1} - v(s)u(s)\tilde{v}_k(s) + v(s)^2u_k(s) - u(s)\tilde{v}_{k+1}(s)\right. \\
& \quad \left.+ u(s)w(s)u_k(s) - u(s)v(s)\tilde{v}_k(s) - v_1(s)u_k(s) + u_1(s)\tilde{v}_k(s)\right) \\
& + \frac{-20c^2 + 3}{2}p(s)u_k(s) + \frac{-20c^2 + 3}{2}q(s)v_k(s) + 20c^2q(s)u(s)u_k(s) \Big] \frac{n^{-\frac{2}{3}}}{20} \\
& \quad + O\left(\frac{1}{n}\right)e_P(s) \Big\}. \tag{2.9}
\end{aligned}$$

In [3], we found an expression for $R_n(x, y) = \rho_n(x, y) \cdot K_{n,2}(x, y)$, this also follows from (1.8) and (2.1). Note that the following representation will give the same result,

$$R_n(x, y; t) = \frac{Q_n(x; t)P_n(y; t) - P_n(x; t)Q_n(y; t)}{x - y} \tag{2.10}$$

$$\begin{aligned}
R_n(\tau(X), \tau(Y); \tau(s))dx &= \left[R(X, Y; s) - c Q(X; s)Q(Y; s) n^{-\frac{1}{3}} \right. \\
& \quad \left. + \frac{n^{-\frac{2}{3}}}{20} \left[P_1(X; s)P(Y; s) + P(X; s)P_1(Y; s) \right. \right. \\
& \quad \left. - Q_2(X; s)Q(Y; s) - Q_1(X; s)Q_1(Y; s) - Q(X; s)Q_2(Y; s) + 20c^2u_0(s)Q(X; s)Q(Y; s) \right. \\
& \quad \left. \left. + \frac{3 - 20c^2}{2} (P(X; s)Q(Y; s) + Q(X; s)P(Y; s)) \right] + O(n^{-1})e_n(X, Y) \right] dX. \tag{2.11}
\end{aligned}$$

3 Epsilon dependent functions

The corresponding epsilon functions come from the study of the leftmost eigenvalue from GOE and GSE. We present here the system of equation satisfied by those functions and a solution to these equations leading to our desired functions.

To simplify notations we define

$$V_{n,\varepsilon}(t) = 1 - \tilde{v}_{n,\varepsilon}(t), \quad \text{and} \quad \tilde{\mathcal{R}}_{n,1}(t) = 1 - \mathcal{R}_{n,1}(t). \tag{3.1}$$

With this notation, system is

$$\frac{d}{dt} \begin{pmatrix} u_{n,\varepsilon}(t) \\ V_{n,\varepsilon}(t) \\ q_{n,\varepsilon}(t) \end{pmatrix} = \begin{pmatrix} 0 & 0 & -q_n(t) \\ 0 & 0 & p_n(t) \\ -p_n(t) & q_n(t) & 0 \end{pmatrix} \cdot \begin{pmatrix} u_{n,\varepsilon}(t) \\ V_{n,\varepsilon}(t) \\ q_{n,\varepsilon}(t) \end{pmatrix}, \tag{3.2}$$

the boundary conditions in this case are

$$\begin{pmatrix} u_{n,\varepsilon}(\infty) \\ V_{n,\varepsilon}(\infty) \\ q_{n,\varepsilon}(\infty) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ c_\varphi \end{pmatrix}. \tag{3.3}$$

For the orthogonal ensemble

$$\frac{d}{dt} \begin{pmatrix} \mathcal{Q}_{n,1}(t) \\ \mathcal{P}_{n,1}(t) \\ \tilde{\mathcal{R}}_{n,1}(t) \end{pmatrix} = \begin{pmatrix} 0 & 0 & q_n(t) \\ 0 & 0 & p_n(t) \\ p_n(t) & q_n(t) & 0 \end{pmatrix} \cdot \begin{pmatrix} \mathcal{Q}_{n,1}(t) \\ \mathcal{P}_{n,1}(t) \\ \tilde{\mathcal{R}}_{n,1}(t) \end{pmatrix}, \quad (3.4)$$

with boundary conditions in this case are

$$\begin{pmatrix} \mathcal{Q}_{n,1}(\infty) \\ \mathcal{P}_{n,1}(\infty) \\ \tilde{\mathcal{R}}_{n,1}(\infty) \end{pmatrix} = \begin{pmatrix} 2c_\varphi \\ 0 \\ 1 \end{pmatrix} \quad \text{as } n \text{ is even.} \quad (3.5)$$

We also have for the symplectic ensemble

$$\frac{d}{dt} \begin{pmatrix} \mathcal{Q}_{n,4}(t) \\ \mathcal{P}_{n,4}(t) \\ \tilde{\mathcal{R}}_{n,4}(t) \end{pmatrix} = \begin{pmatrix} 0 & 0 & -q_n(t) \\ 0 & 0 & -p_n(t) \\ -p_n(t) & -q_n(t) & 0 \end{pmatrix} \cdot \begin{pmatrix} \mathcal{Q}_{n,4}(t) \\ \mathcal{P}_{n,4}(t) \\ \tilde{\mathcal{R}}_{n,4}(t) \end{pmatrix}. \quad (3.6)$$

where $\tilde{\mathcal{R}}_{n,4}(t) = 1 + \mathcal{R}_{n,4}(t)$, with corresponding boundary conditions

$$\begin{pmatrix} \mathcal{Q}_{n,4}(\infty) \\ \mathcal{P}_{n,4}(\infty) \\ \tilde{\mathcal{R}}_{n,4}(\infty) \end{pmatrix} = \begin{pmatrix} -c_\varphi \\ -c_\psi \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -c_\psi \\ 1 \end{pmatrix} \quad \text{as } n \text{ is odd.} \quad (3.7)$$

The first two set of equations were solved in [4], here we give the general solution from the series expansion derived there. We will not go back into the derivation, but would like to point out that this is the direct consequence of those matrix exponentials. Our goal here is to give a close formula for those functions.

We define

$$a(t) = \int_t^\infty q_n(x) dx \quad \text{and} \quad b(t) = \int_t^\infty p_n(x) dx. \quad (3.8)$$

We note that these two functions scale (under the transformation τ) in the large n limit to the same function

$$\frac{1}{\sqrt{2}} \int_s^\infty q(x) dx = \frac{1}{\sqrt{2}} \mu(s).$$

We give this to justify our notation used bellow, and it says that for very large n , the argument of all the hyperbolic functions is real. With this notation, we have

$$u_{n,\varepsilon}(t) = \frac{a(t)}{2b(t)} [1 - \cosh \sqrt{2a(t)b(t)}] + c_\varphi \sqrt{\frac{a(t)}{2b(t)}} \sinh \sqrt{2a(t)b(t)}, \quad (3.9)$$

$$\tilde{V}_{n,\varepsilon}(t) = \frac{1}{2} [1 + \cosh \sqrt{2a(t)b(t)}] - c_\varphi \sqrt{\frac{b(t)}{2a(t)}} \sinh \sqrt{2a(t)b(t)}, \quad (3.10)$$

or

$$\tilde{v}_{n,\varepsilon}(t) = 1 - \frac{1}{2} [1 + \cosh \sqrt{2a(t)b(t)}] + c_\varphi \sqrt{\frac{b(t)}{2a(t)}} \sinh \sqrt{2a(t)b(t)} \quad (3.11)$$

and

$$q_{n,\varepsilon}(t) = -\sqrt{\frac{a(t)}{2b(t)}} \sinh \sqrt{2a(t)b(t)} + c_\varphi \cosh \sqrt{2a(t)b(t)}. \quad (3.12)$$

This result is valid¹ for the GOE when $c_\varphi \neq 0$ and for the GSE we have $c_\varphi = 0$

In the same way we find that for the GOE case, the calligraphic functions are

$$\mathcal{Q}_{n,1}(t) = c_\varphi [1 + \cosh \sqrt{2a(t)b(t)}] - \sqrt{\frac{a(t)}{2b(t)}} \sinh \sqrt{2a(t)b(t)}, \quad (3.13)$$

$$\mathcal{P}_{n,1}(t) = c_\varphi \frac{b(t)}{a(t)} [\cosh \sqrt{2a(t)b(t)} - 1] - \sqrt{\frac{b(t)}{2a(t)}} \sinh \sqrt{2a(t)b(t)}, \quad (3.14)$$

and

$$\tilde{\mathcal{R}}_{n,1}(t) = -2c_\varphi \sqrt{\frac{b(t)}{2a(t)}} \sinh \sqrt{2a(t)b(t)} + \cosh \sqrt{2a(t)b(t)} \quad (3.15)$$

or

$$\mathcal{R}_{n,1}(t) = 1 + 2c_\varphi \sqrt{\frac{b(t)}{2a(t)}} \sinh \sqrt{2a(t)b(t)} - \cosh \sqrt{2a(t)b(t)} \quad (3.16)$$

where

$$c_\varphi = (\pi n)^{1/4} 2^{-3/4-n/2} \frac{(n!)^{1/2}}{(n/2)!}. \quad (3.17)$$

A large n expansion for $v_{n,\varepsilon}$, $q_{n,\varepsilon}$ is given in [4] on page 17. A large n expansion of $\mathcal{P}_{n,1}$ is equation (3.58) and $\mathcal{R}_{n,1}$ is equation (3.59) of the same work. We will therefore give here an expression for $u_{n,\varepsilon}$ and $\mathcal{Q}_{n,1}$ for large n . Substitution of $a(t) = \int_t^\infty q_n(x) dx$ and $b(t) = \int_t^\infty p_n(x) dx$ into (3.9) and (3.13) yields the following results.

Theorem 3.1. *for s bounded away from minus infinity,*

$$\begin{aligned} u_{n,\varepsilon}(\tau(s)) &= \frac{1}{2}(1 - e^{-\mu(s)}) + \left(\frac{\nu(s)}{4\mu(s)}(e^{-\mu(s)} + \cosh(\mu(s)) - 2) - \frac{cq(s)}{2}e^{-\mu(s)} \right) n^{-\frac{1}{3}} + \\ &\quad \frac{1}{32\mu(s)^2} \left(e^{-\mu(s)} (\nu(s)^2 (-(-1 + e^{\mu(s)}) (-5 - 12c + (3 + 4c)e^{\mu(s)})) - \right. \\ &\quad \left. 2\mu(s) (1 + 6c - 2ce^{2\mu(s)} + 4c^2\mu(s))) + \right. \\ &\quad \left. 4c\nu(s) (3 - 4e^{\mu(s)} - e^{2\mu(s)}(-1 + \mu(s)) + 3\mu(s) + 4c\mu(s)^2) \int_s^\infty q[x]u[x] dx + \right. \\ &\quad \left. 8\mu(s) \left(-10c(-3 + e^{\mu(s)}) (-1 + e^{\mu(s)}) \left(\int_s^\infty q[x]v[x] dx - \int_s^\infty q_1[x] dx \right) + \right. \right. \\ &\quad \left. \left. \mu(s) \left((3 - 20c^2) \int_s^\infty p[x]u[x] dx + 3 \int_s^\infty q[x]v[x] dx + \right. \right. \right. \end{aligned}$$

¹The computation for GOE assumes n to be even and for GSE assumes n to be odd

$$\begin{aligned}
& 2 \int_s^\infty v[x]p_1[x] dx + 2 \int_s^\infty p_2[x] dx + 3 \int_s^\infty q_1[x] dx - c^2 \left(\left(\int_s^\infty q[x]u[x] dx \right)^2 - \right. \\
& 20 \left(2 \int_s^\infty q[x]u[x]^2 dx - 3 \int_s^\infty q[x]v[x] dx + \int_s^\infty q_1[x] dx \right) \left. \right) - 2 \left(\int_s^\infty u[x]q_2[x] dx + \right. \\
& \left. \int_s^\infty q_1[x]u_1[x] dx + \int_s^\infty q[x]u_2[x] dx - \int_s^\infty p[x]v_1[x] dx \right) \left. \right) \Big) n^{-\frac{2}{3}} + O(n^{-1})
\end{aligned}$$

We also have

$$\begin{aligned}
\tilde{v}_{n,\varepsilon}(\tau(s)) &= \frac{1}{2} (1 - e^{-\mu(s)}) + \left(\frac{\nu(s)}{4\mu(s)} \sinh(\mu(s)) + \frac{cq(s)}{2} e^{-\mu(s)} \right) n^{-\frac{1}{3}} \\
& \frac{1}{16\mu(s)^2} \left\{ \left(4c\nu(s) \int_s^\infty q[x]u[x] dx \left(-\cosh[\mu(s)]\mu(s) + 2ce^{-\mu(s)}\mu(s)^2 + \sinh[\mu(s)] \right) \right. \right. \\
& + \nu(s)^2 (\cosh[\mu(s)]\mu(s) (-1 + 4c - 4c^2\mu(s)) + (1 - 4c + \mu(s) + 4c^2\mu(s)^2) \sinh[\mu(s)]) - \\
& 4\mu(s) \left(e^{-\mu(s)}\mu(s) \left((-3 + 20c^2) \int_s^\infty p[x]u[x] dx - 3 \int_s^\infty q[x]v[x] dx - 2 \int_s^\infty v[x]p_1[x] dx \right. \right. \\
& - 2 \int_s^\infty p_2[x] dx - 3 \int_s^\infty q_1[x] dx + c^2 \left(\left(\int_s^\infty q[x]u[x] dx \right)^2 - 20 \left(2 \int_s^\infty q[x]u[x]^2 dx \right. \right. \\
& - 3 \int_s^\infty q[x]v[x] dx + \int_s^\infty q_1[x] dx \right) \left. \right) + 2 \left(\int_s^\infty u[x]q_2[x] dx + \int_s^\infty q_1[x]u_1[x] dx + \right. \\
& \left. \int_s^\infty q[x]u_2[x] dx - \int_s^\infty p[x]v_1[x] dx \right) \left. \right) + 20c \left(\int_s^\infty q[x]v[x] dx - \int_s^\infty q_1[x] dx \right) \sinh[\mu(s)] \Big\} n^{-\frac{2}{3}} \\
& + O(n^{-1})
\end{aligned}$$

we also have

$$\begin{aligned}
q_{n,\varepsilon}(\tau(s)) &= \frac{e^{-\mu(s)}}{\sqrt{2}} + \left(\frac{\nu(s)}{2\sqrt{2}\mu(s)} \sinh \mu(s) + \frac{cq(s)}{\sqrt{2}} e^{-\mu(s)} \right) n^{-\frac{1}{3}} \\
& \left\{ \frac{1}{8\sqrt{2}\mu(s)^2} \left(\left(-4c\nu(s) \left(\int_s^\infty q[x]u[x] dx \right) \left(\mu(s) (\cosh[\mu(s)] + 2ce^{-\mu(s)}\mu(s)) - \sinh[\mu(s)] \right) \right. \right. \right. \\
& + \nu(s)^2 (\cosh[\mu(s)]\mu(s) (1 + 4c + 4c^2\mu(s)) - (1 + 4c + \mu(s) + 4c^2\mu(s)^2) \sinh[\mu(s)]) + \\
& 4\mu(s) \left(e^{-\mu(s)}\mu(s) \left((-3 + 20c^2) \int_s^\infty p[x]u[x] dx - 3 \int_s^\infty q[x]v[x] dx - 2 \int_s^\infty v[x]p_1[x] dx - \right. \right. \\
& 2 \int_s^\infty p_2[x] dx - 3 \int_s^\infty q_1[x] dx + c^2 \left(\left(\int_s^\infty q[x]u[x] dx \right)^2 - 20 \left(2 \int_s^\infty q[x]u[x]^2 dx - \right. \right. \\
& 3 \int_s^\infty q[x]v[x] dx + \int_s^\infty q_1[x] dx \right) \left. \right) + 2 \left(\int_s^\infty u[x]q_2[x] dx + \int_s^\infty q_1[x]u_1[x] dx \right. \\
& \left. \left. + \int_s^\infty q[x]u_2[x] dx - \int_s^\infty p[x]v_1[x] dx \right) \right) \Big\} +
\end{aligned}$$

$$20c \left(- \int_s^\infty q[x]v[x] dx + \int_s^\infty q_1[x] dx \right) \sinh[\mu(s)] \Big) \Big\} n^{-\frac{2}{3}}$$

and the GOE_n calligraphic variables are

Theorem 3.2. *for s bounded away from minus infinity,*

$$\begin{aligned} \mathcal{Q}_{n,1}(\tau(s)) &= \frac{1}{\sqrt{2}}(1 + e^{-\mu(s)}) + \left(\frac{\nu(s)}{2\sqrt{2}\mu(s)} \sinh(\mu(s)) + \frac{cq(s)}{\sqrt{2}} e^{-\mu(s)} \right) n^{-\frac{1}{3}} + \\ &\frac{1}{8\sqrt{2}\mu(s)^2} \left(\left(-4c\nu(s) \int_s^\infty q[x]u[x] dx (\mu(s) (\cosh[\mu(s)] + 2ce^{-\mu(s)}\mu(s)) - \right. \right. \\ &\sinh \mu(s)) + \nu(s)^2 (\cosh[\mu(s)]\mu(s) (1 + 4c + 4c^2\mu(s)) - (1 + 4c + \mu(s) + 4c^2\mu(s)^2) \sinh \mu(s)) + \\ &4\mu(s) \left(e^{-\mu(s)}\mu(s) \left((-3 + 20c^2) \int_s^\infty p[x]u[x] dx - 3 \int_s^\infty q[x]v[x] dx - 2 \int_s^\infty v[x]p_1[x] dx \right. \right. \\ &- 2 \int_s^\infty p_2[x] dx - 3 \int_s^\infty q_1[x] dx + c^2 \left(\left(\int_s^\infty q[x]u[x] dx \right)^2 - 20 \left(2 \int_s^\infty q[x]u[x]^2 dx \right. \right. \\ &- 3 \int_s^\infty q[x]v[x] dx + \int_s^\infty q_1[x] dx \right) \Big) + 2 \left(\int_s^\infty u[x]q_2[x] dx + \int_s^\infty q_1[x]u_1[x] dx + \right. \\ &\left. \left. \int_s^\infty q[x]u_2[x] dx - \int_s^\infty p[x]v_1[x] dx \right) \right) + 20c \left(\int_s^\infty q_1[x] dx - \int_s^\infty q[x]v[x] dx \right) \sinh \mu(s) \Big) \Big) n^{-\frac{2}{3}} \\ &+ O(n^{-1}), \end{aligned}$$

we also have

$$\begin{aligned} \mathcal{P}_{n,1}(\tau(s)) &= \left\{ \frac{-1 + e^{-\mu(s)}}{\sqrt{2}} + \left(\frac{\nu(s)}{2\sqrt{2}\mu(s)} (e^{-\mu(s)} + \cosh \mu(s)) + \frac{cq(s)}{\sqrt{2}} e^{-\mu(s)} \right) n^{-\frac{1}{3}} \right. \\ &- \frac{1}{16(\sqrt{2}\mu(s)^2)} \left((e^{-\mu(s)} (\nu(s)^2 ((-1 + e^{\mu(s)}) (5 - 12c + (-3 + 4c)e^{\mu(s)}) - \right. \\ &2(\mu(s)) (1 + 2c(e^{2\mu(s)} - 3) + 4c^2\mu(s))) + 4c\nu(s) (4e^{\mu(s)} - 3 + e^{2\mu(s)}(\mu(s) - 1) - 3\mu(s) + 4c\mu(s)^2) \\ &\int_s^\infty q[x]u[x] dx + 8\mu(s) (10c(-3 + e^{\mu(s)}) (-1 + e^{\mu(s)}) \left(\int_s^\infty q[x]v[x] dx - \int_s^\infty q_1[x] dx \right) + \\ &\mu(s) \left((3 - 20c^2) \int_s^\infty p[x]u[x] dx + 3 \int_s^\infty q[x]v[x] dx + 2 \int_s^\infty v[x]p_1[x] dx + 2 \int_s^\infty p_2[x] dx + \right. \\ &3 \int_s^\infty q_1[x] dx - c^2 \left(\left(\int_s^\infty q[x]u[x] dx \right)^2 - 20 \left(2 \int_s^\infty q[x]u[x]^2 dx - 3 \int_s^\infty q[x]v[x] dx + \right. \right. \\ &\left. \left. \int_s^\infty q_1[x] dx \right) \right) \Big) - \\ &\left. \left. 2 \left(\int_s^\infty u[x]q_2[x] dx + \int_s^\infty q_1[x]u_1[x] dx + \int_s^\infty q[x]u_2[x] dx - \int_s^\infty p[x]v_1[x] dx \right) \right) \right) \Big) \Big) n^{-\frac{2}{3}} \end{aligned}$$

$$+O(n^{-1})$$

and the last of the GOE_n function is

$$\begin{aligned} \mathcal{R}_{n,1}(\tau(s)) = & (1 - e^{-\mu(s)}) + \left(\frac{\nu(s)}{2\mu(s)} \sinh \mu(s) - cq(s)e^{-\mu(s)} \right) n^{-\frac{1}{3}} + \\ & \frac{1}{8\mu(s)^2} ((4c\nu(s)\mu(s) (-\cosh[\mu(s)]\mu(s) + 2ce^{-\mu(s)}\mu(s)^2 + \sinh[\mu(s)]) + \\ & \nu(s)^2(\cosh[\mu(s)]\mu(s) (-1 + 4c - 4c^2\mu(s)) + (1 - 4c + \mu(s) + 4c^2\mu(s)^2) \sinh[\mu(s)]) - \\ & 4\mu(s) \left(e^{-\mu(s)}\mu(s) \left((-3 + 20c^2) \int_s^\infty p[x]u[x] dx - 3 \int_s^\infty q[x]v[x] dx - 2 \int_s^\infty v[x]p_1[x] dx \right. \right. \\ & - 2 \int_s^\infty p_2[x] dx - 3 \int_s^\infty q_1[x] dx + c^2 \left(\left(\int_s^\infty q[x]u[x] dx \right)^2 - 20 \left(2 \int_s^\infty q[x]u[x]^2 dx \right. \right. \\ & - 3 \int_s^\infty q[x]v[x] dx + \int_s^\infty q_1[x] dx \right) \left. \right) + 2 \left(\int_s^\infty u[x]q_2[x] dx + \int_s^\infty q_1[x]u_1[x] dx + \right. \\ & \left. \left. \int_s^\infty q[x]u_2[x] dx - \int_s^\infty p[x]v_1[x] dx \right) \right) + \\ & \left. 20c \left(\int_s^\infty q[x]v[x] dx - \int_s^\infty q_1[x] dx \right) \sinh[\mu(s)] \right) n^{-\frac{2}{3}} + O(n^{-1}) \end{aligned}$$

For the symplectic ensemble, the calligraphic variables were mentioned in [4], the similarity of the corresponding system of differential equations to the GOE calligraphic system makes our derivation simpler, the coefficient matrices are the same up the minus sign. Keeping with the same notation, we see that

$$\begin{pmatrix} \mathcal{Q}_{n,4}(t) \\ \mathcal{P}_{n,4}(t) \\ \tilde{\mathcal{R}}_{n,4}(t) \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(1 + \cosh \sqrt{2ab}) & \frac{a}{2b}(\cosh \sqrt{2ab} - 1) & \sqrt{\frac{a}{2b}} \sinh \sqrt{2ab} \\ \frac{b}{2a}(\cosh \sqrt{2ab} - 1) & \frac{1}{2}(1 + \cosh \sqrt{2ab}) & \sqrt{\frac{b}{2a}} \sinh \sqrt{2ab} \\ \sqrt{\frac{b}{2a}} \sinh \sqrt{2ab} & \sqrt{\frac{a}{2b}} \sinh \sqrt{2ab} & \cosh \sqrt{2ab} \end{pmatrix} \cdot \begin{pmatrix} 0 \\ -c_\psi \\ 1 \end{pmatrix}. \quad (3.18)$$

We dropped the t dependence of a and b in the above matrix for esthetic reason. This gives

$$\mathcal{Q}_{n,4}(t) = -c_\psi \frac{a(t)}{2b(t)} [\cosh \sqrt{2a(t)b(t)} - 1] - \sqrt{\frac{a(t)}{2b(t)}} \sinh \sqrt{2a(t)b(t)}, \quad (3.19)$$

$$\mathcal{P}_{n,4}(t) = -c_\psi \frac{1}{2} [1 + \cosh \sqrt{2a(t)b(t)}] - \sqrt{\frac{b(t)}{2a(t)}} \sinh \sqrt{2a(t)b(t)}, \quad (3.20)$$

and

$$\tilde{\mathcal{R}}_{n,4}(t) = -c_\psi \sqrt{\frac{a(t)}{2b(t)}} \sinh \sqrt{2a(t)b(t)} + \cosh \sqrt{2a(t)b(t)} \quad (3.21)$$

or

$$\mathcal{R}_{n,4}(t) = -c_\psi \sqrt{\frac{a(t)}{2b(t)}} \sinh \sqrt{2a(t)b(t)} + \cosh \sqrt{2a(t)b(t)} - 1. \quad (3.22)$$

for the GSE, we have the corresponding formula for $u_{n,\varepsilon}$, $\tilde{v}_{n,\varepsilon}$ and $q_{n,\varepsilon}$.

Theorem 3.3. *for s bounded away from minus infinity,*

$$\begin{aligned} u_{n,\varepsilon}(\tau(s)) = & -\sinh^2\left(\frac{\mu(s)}{2}\right) + \left(\frac{\nu(s)}{2\mu(s)}(\cosh(\mu(s)) - 1) - \frac{cq(s)}{2}\sinh(\mu(s))\right)n^{-\frac{1}{3}} + \\ & \frac{1}{16\mu(s)^2} \left(\left(8c\nu(s) \int_s^\infty q[x]u[x] dx (-1 + \cosh[\mu(s)]) (1 + c\mu(s)^2) - \right. \right. \\ & \mu(s) \sinh[\mu(s)] + \nu(s)^2(4+8c-4\cosh[\mu(s)]) (1 + 2c + c^2\mu(s)^2) + (1 + 8c)\mu(s) \sinh[\mu(s)] \Big) + \\ & 4\mu(s) \left(40c(-1 + \cosh[\mu(s)]) \left(-\int_s^\infty q[x]v[x] dx + \int_s^\infty q_1[x] dx \right) + \right. \\ & \mu(s) \left(-c^2 \cosh[\mu(s)] \left(\int_s^\infty q[x]u[x] dx \right)^2 + \left((-3 + 20c^2) \int_s^\infty p[x]u[x] dx - \right. \right. \\ & 3 \int_s^\infty q[x]v[x] dx - 2 \int_s^\infty v[x]p_1[x] dx - 2 \int_s^\infty p_2[x] dx - 3 \int_s^\infty q_1[x] dx - \\ & 20c^2 \left(2 \int_s^\infty q[x]u[x]^2 dx - 3 \int_s^\infty q[x]v[x] dx + \int_s^\infty q_1[x] dx \right) + 2 \left(\int_s^\infty u[x]q_2[x] dx + \right. \\ & \left. \left. \int_s^\infty q_1[x]u_1[x] dx + \int_s^\infty q[x]u_2[x] dx - \int_s^\infty p[x]v_1[x] dx \right) \sinh[\mu(s)] \right) \Big) \Big) n^{-\frac{2}{3}} \\ & + O(n^{-1}) \end{aligned}$$

and

$$\begin{aligned} \tilde{v}_{n,\varepsilon}(\tau(s)) = & -\sinh^2\left(\frac{\mu(s)}{2}\right) + \frac{cq(s)}{2}\sinh(\mu(s))n^{-\frac{1}{3}} + \\ & \frac{1}{16\mu(s)} \left(\left(-4c^2 \cosh[\mu(s)]\mu(s)(\nu(s) - \int_s^\infty q[x]u[x] dx)^2 + \right. \right. \\ & \left(\nu(s)^2 + 4\mu(s) \left((-3 + 20c^2) \int_s^\infty p[x]u[x] dx - 3 \int_s^\infty q[x]v[x] dx - \right. \right. \\ & 2 \int_s^\infty v[x]p_1[x] dx - 2 \int_s^\infty p_2[x] dx - 3 \int_s^\infty q_1[x] dx - 20c^2 \left(2 \int_s^\infty q[x]u[x]^2 dx - \right. \\ & \left. \left. 3 \int_s^\infty q[x]v[x] dx + \int_s^\infty q_1[x] dx \right) + 2 \left(\int_s^\infty u[x]q_2[x] dx + \int_s^\infty q_1[x]u_1[x] dx + \right. \right. \end{aligned}$$

$$\int_s^\infty q[x]u_2[x] dx - \int_s^\infty p[x]v_1[x] dx \Big) \Big) \sinh[\mu(s)] \Big) n^{-\frac{2}{3}} + O(n^{-1})$$

we also have

$$\begin{aligned} q_{n,\varepsilon}(\tau(s)) = & -\frac{1}{\sqrt{2}} \sinh(\mu(s)) + \left(\frac{\nu(s)}{2\sqrt{2}\mu(s)} \sinh(\mu(s)) + \frac{cq(s)}{\sqrt{2}} \cosh(\mu(s)) \right) n^{-\frac{1}{3}} + \\ & \frac{1}{8\sqrt{2}\mu(s)^2} \left(\left(\cosh[\mu(s)]\mu(s) \left((1+4c)\nu(s)^2 - 4c\nu(s) \int_s^\infty q[x]u[x] dx + \right. \right. \right. \\ & 4\mu(s) \left((-3+20c^2) \int_s^\infty p[x]u[x] dx - 3 \int_s^\infty q[x]v[x] dx - 2 \int_s^\infty v[x]p_1[x] dx - 2 \right. \\ & \left. \int_s^\infty p_2[x] dx - 3 \int_s^\infty q_1[x] dx - 20c^2 \left(2 \int_s^\infty q[x]u[x]^2 dx - 3 \int_s^\infty q[x]v[x] dx + \int_s^\infty q_1[x] dx \right) + \right. \\ & \left. \left. 2 \left(\int_s^\infty u[x]q_2[x] dx + \int_s^\infty q_1[x]u_1[x] dx + \int_s^\infty q[x]u_2[x] dx - \int_s^\infty p[x]v_1[x] dx \right) \right) \right) - \\ & \left(\nu(s)^2 (1+4c+4c^2\mu(s)^2) - 4c\nu(s) (1+2c\mu(s)^2) \int_s^\infty q[x]u[x] dx + \right. \\ & \left. 4c\mu(s) \left(c\mu(s) \left(\int_s^\infty q[x]u[x] dx \right)^2 + 20 \left(\int_s^\infty q[x]v[x] dx - \int_s^\infty q_1[x] dx \right) \right) \right) \sinh[\mu(s)] n^{-\frac{2}{3}} \\ & + O(n^{-1}). \end{aligned}$$

For the GSE Calligraphic functions we have the following expansions,

Theorem 3.4. *for s bounded away from minus infinity,*

$$\begin{aligned} \mathcal{Q}_{n,4}(\tau(s)) = & \frac{1}{2\sqrt{2}} (1 - \cosh \mu(s) + 2 \sinh \mu(s)) + \\ & \left(\frac{\nu(s)}{2\sqrt{2}\mu(s)} (e^{\mu(s)} - 1) - \frac{cq(s)}{2\sqrt{2}} (2 \cosh \mu(s) - \sinh \mu(s)) \right) n^{-\frac{1}{3}} + \\ & \frac{1}{32\sqrt{2}\mu(s)^2} (e^{-\mu(s)} (\nu(s)^2 (-2(-1+e^{\mu(s)})(-3-8c+e^{\mu(s)}) - \\ & (3+16c+e^{2\mu(s)})\mu(s) + 4c^2(-3+e^{2\mu(s)})\mu(s)^2) - 8c\nu(s) \\ & (2(-1+e^{\mu(s)}) + \mu(s)(-2+c(-3+e^{2\mu(s)})\mu(s))) \int_s^\infty q[x]u[x] dx + \\ & 4\mu(s) \left(80c(-1+e^{\mu(s)}) \left(\int_s^\infty q[x]v[x] dx - \int_s^\infty q_1[x] dx \right) + \mu(s) \right. \\ & \left. \left(-(-3+20c^2)(3+e^{2\mu(s)}) \int_s^\infty p[x]u[x] dx + c^2(-3+e^{2\mu(s)}) \left(\int_s^\infty q[x]u[x] dx \right)^2 + \right. \right. \\ & \left. \left. (3+e^{2\mu(s)}) \left(40c^2 \int_s^\infty q[x]u[x]^2 dx + (3-60c^2) \int_s^\infty q[x]v[x] dx + \right. \right. \right. \end{aligned}$$

$$2 \int_s^\infty v[x]p_1[x] dx + 2 \int_s^\infty p_2[x] dx + (3 + 20c^2) \int_s^\infty q_1[x] dx - 2 \left(\int_s^\infty u[x]q_2[x] dx + \int_s^\infty q_1[x]u_1[x] dx + \int_s^\infty q[x]u_2[x] dx - \int_s^\infty p[x]v_1[x] dx \right) \Big) \Big) \Big) \Big) n^{-\frac{2}{3}} + O(n^{-1})$$

the next function is

$$\begin{aligned} \mathcal{P}_{n,4}(\tau(s)) &= \frac{1}{2\sqrt{2}}(2 \sinh \mu(s) - \cosh \mu(s) - 1) \\ &+ \left(\frac{\nu(s)}{2\sqrt{2}\mu(s)} \sinh \mu(s) - \frac{cq(s)}{2\sqrt{2}} (2 \cosh \mu(s) - \sinh \mu(s)) \right) n^{-\frac{1}{3}} \\ &\frac{1}{16\sqrt{2}\mu(s)^2} ((8c\nu(s) \int_s^\infty q[x]u[x] dx (-\cosh[\mu(s)]\mu(s) + \\ &c\mu(s)^2(\cosh[\mu(s)] - 2 \sinh[\mu(s)]) + \sinh[\mu(s)]) + \\ &\nu(s)^2 (-2 \cosh[\mu(s)]\mu(s) (1 - 4c + 2c^2\mu(s)) + \\ &(2 - 8c + \mu(s) + 8c^2\mu(s)^2) \sinh[\mu(s)]) - \\ &4\mu(s) \left(\mu(s) \left(c^2 \left(\int_s^\infty q[x]u[x] dx \right)^2 (\cosh[\mu(s)] - 2 \sinh[\mu(s)]) + \right. \right. \\ &\left. \left. (-3 + 20c^2) \left(\int_s^\infty p[x]u[x] dx \right) (2 \cosh[\mu(s)] - \sinh[\mu(s)]) - \right. \right. \\ &\left. \left. \left(40c^2 \int_s^\infty q[x]u[x]^2 dx + (3 - 60c^2) \int_s^\infty q[x]v[x] dx + 2 \int_s^\infty v[x]p_1[x] dx + \right. \right. \\ &\left. \left. 2 \int_s^\infty p_2[x] dx + (3 + 20c^2) \int_s^\infty q_1[x] dx - 2 \left(\int_s^\infty u[x]q_2[x] dx + \int_s^\infty q_1[x]u_1[x] dx + \right. \right. \right. \\ &\left. \left. \left. \int_s^\infty q[x]u_2[x] dx - \int_s^\infty p[x]v_1[x] dx \right) \right) (2 \cosh[\mu(s)] - \sinh[\mu(s)]) \right) + \\ &\left. 40c \left(\int_s^\infty q[x]v[x] dx - \int_s^\infty q_1[x] dx \right) \sinh[\mu(s)] \right) \Big) \Big) n^{-\frac{2}{3}} + O(n^{-1}) \end{aligned}$$

the last of our function is

$$\begin{aligned} \mathcal{R}_{n,4}(\tau(s)) &= \left(\cosh \mu(s) - \frac{1}{2} \sinh \mu(s) - 1 \right) + \\ &\left(\frac{\nu(s)}{4\mu(s)} \sinh \mu(s) + \frac{cq(s)}{2} (\cosh \mu(s) - 2 \sinh \mu(s)) \right) n^{-\frac{1}{3}} \\ &\frac{1}{16\mu(s)^2} ((-4c\nu(s) \left(\int_s^\infty q[x]u[x] dx \right) (\cosh[\mu(s)]\mu(s) \\ &(1 + 4c\mu(s)) - (1 + 2c\mu(s)^2) \sinh[\mu(s)]) + \\ &\nu(s)^2 (\cosh[\mu(s)]\mu(s) (1 + 4c + 8c^2\mu(s)) - \end{aligned}$$

$$\begin{aligned}
& (1 + 4c + 2\mu(s) + 4c^2\mu(s)^2) \sinh[\mu(s)] + 4\mu(s) \\
& \left(\mu(s) \left((-3 + 20c^2) \int_s^\infty p[x]u[x] dx (\cosh[\mu(s)] - 2\sinh[\mu(s)]) - \right. \right. \\
& \left(40c^2 \int_s^\infty q[x]u[x]^2 dx + (3 - 60c^2) \int_s^\infty q[x]v[x] dx + 2 \int_s^\infty v[x]p_1[x] dx + \right. \\
& 2 \int_s^\infty p_2[x] dx + (3 + 20c^2) \int_s^\infty q_1[x] dx - 2 \left(\int_s^\infty u[x]q_2[x] dx + \int_s^\infty q_1[x]u_1[x] dx + \right. \\
& \left. \left. \int_s^\infty q[x]u_2[x] dx - \int_s^\infty p[x]v_1[x] dx \right) \right) (\cosh[\mu(s)] - 2\sinh[\mu(s)]) + \\
& \left. c^2 \left(\int_s^\infty q[x]u[x] dx \right)^2 (2\cosh[\mu(s)] - \sinh[\mu(s)]) \right) + \\
& 20c \left(- \int_s^\infty q[x]v[x] dx + \int_s^\infty q_1[x] dx \right) \sinh[\mu(s)] \Big) n^{-\frac{2}{3}} + O(n^{-1})
\end{aligned}$$

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